ground water

Methods Note/

On Simulation and Analysis of Variable-Rate Pumping Tests

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Abstract

Analytical solutions for constant-rate pumping tests are widely used to infer aquifer properties. In this note, we implement a methodology that approximates the time-varying pumping record as a series of segments with linearly varying pumping rates. We validate our approach using an analytical solution for a sinusoidally varying pumping test. We also apply our methodology to analyze synthetic test data and compare the results with those from a commonly used method where rate variations are represented by a series of constant-rate steps.

Introduction

Hydraulic properties of an aquifer are commonly inferred by fitting drawdown and/or recovery data recorded from pumping tests to analytical solutions for radial flow toward a pumping well. For mathematical simplicity, such analytical solutions are commonly derived for constant-rate conditions. However, the pumping rate may vary either intentionally or because of technical difficulties during the test.

The most common approach to analyze pumping tests with variable pumping rates is based on superposition of piecewise constant rates. Considering a confined aquifer as a linear system with time-invariant boundary conditions, Cooper and Jacob (1946) applied the superposition principle to account for stepwise changes in pumping rates. Abu-Zeid and Scott (1963), Abu-Zeid et al. (1964) and Hantush (1964) proposed analytical solutions for variable-rate pumping tests assuming exponentially decreasing pumping rates. Lai et al. (1973) and Lai and Su (1974) extended the solution of Papadopulos and Cooper (1967) to include leakage from the semi-confining layers when the pumping rates are exponentially and linearly varying. Black and Kipp (1981) provided a solution to an aquifer borehole test for sinusoidal perturbation in a confined non-leaky aquifer. Rasmussen et al. (2003) extended the Hantush (1964) solution to include sinusoidal variation of pumping rates.

In some field applications, the pumping rates are varied intentionally. Butler and McElwee (1990) suggested that variable pumping rates can be used to increase the sensitivity of parameters to observed drawdown, and hence improve parameter identifiably; each time the pumping rate is increased, a new cone of depression (superimposed upon the original one) propagates out from the pumping well, producing an increase in sensitivity and a new interval of time during which the aquifer zone influences drawdown.

Adequate representation of variable pumping rates can be important when various natural phenomena unaccounted for in the analytical solution are causing transients in the observed drawdown records (e.g., barometric effects, infiltration events). In these cases, the analysis of the observed drawdown transients is difficult, if transients caused by variable pumping rates are not accurately captured.

The commonly used approach of constant-rate step changes to represent pumping variability may not always

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be sufficient to capture important details in the observed drawdown transients. For cases where the pumping variability has increasing or decreasing linear trends the method of step changes is generally not suitable unless a large number of closely spaced step changes are introduced. Piecewise-linear approximation of pumping rate variation has been reported in the petroleum engineering literature (Kulpin and Miasnikov 1974; Stewart et al. 1983; Streltsova 1987). Streltsova (1987) presented drawdown due to a variable pumping rate as a summation of drawdown responses due to a series of linear pumping rate segments. However their solution is limited to fully penetrating wells of zero radius (Theis solution). Roumboutsos and Stewart (1988) proposed deconvolving well-test data by numerically transforming the discrete pressure and pumping rate data into the Laplace transform domain. For a variable pumping rate they adopted a piecewise-linear approximation.

In this note, we generalize the approach of Roumboutsos and Stewart (1988) to commonly used analytical solutions for pumping test analysis. We also propose use of the discrete time convolution integral to avoid numerical instabilities that may arise as a result of abrupt changes in pumping rate. We implement this approach in WELLS (available at http://wells.lanl.gov), a computer program designed to analyze multi-well variable-rate pumping tests in confined, unconfined and leaky aquifers in a finite or infinite domain using a variety of analytical solutions. The methodology of piecewise linearly varying pumping rate is demonstrated using an existing analytical solution for confined aquifers, but is also applicable to solutions for unconfined and/or leaky aquifers. After validating our methodology, we analyze synthetic pumping test data by inversely estimating the parameters and compare our results with those for the commonly used method of constant-rate step changes in pumping rate.

Analytical Solution for Variable Pumping Rate

Consider a partially penetrating well of small radius (i.e., $r_\mathrm{w} \to 0$) that is in confined aquifer screened from depths d through l below the aquifer top. For the case when the pumping well is discharging at constant rate Q, the Laplace transformed drawdown $\bar{s}(r, z, t)$ can be expressed as (Hantush 1964):

$$\bar{s}(r_D, z_D, p_D) = \frac{Q}{p} f(r_D, z_D, p_D) = \frac{Q/p}{4\pi T} \begin{cases} 2K_0(\phi_0) + \frac{4}{\pi} \times \\ \sum_{n=1}^{\infty} \frac{K_0(\phi_0)[\sin(n\pi l_D) - \sin(n\pi d_D)]}{n(l_D - d_D)} \cos(n\pi z_D) \end{cases} \tag{1}$$

where p is the Laplace transformation parameter, $r_D = r/b$, $T = K_r b$, $z_D = z/b$, $p_D = pt$, $d_D = d/b$, $l_D = l/b$, $\phi_n = \sqrt{p_D/t_D + \beta^2 n^2 \pi^2}$, $t_D = \alpha_s t/r^2$, $\alpha_s = K_r/S_s$, S_s is specific storage, K_r is horizontal hydraulic conductivity, $\beta = r_D K_D^{1/2}$, K_D is anisotropic ratio, b is the aquifer thickness, and K_0 and K_1 are the second-kind modified Bessel functions of order zero and one, respectively.

The Hantush (1964) solution has the form $\bar{s} = \frac{Q}{p} f(r_D, z_D, p_D)$, where Q/p is the Laplace transform of the constant pumping rate Q; many other constant-rate analytical solutions for Laplace transformed drawdown have similar form. For variable pumping rate $Q(t)$ with Laplace transform $\bar{Q}(p)$, the existing solutions can be directly used by replacing the Q/p with $\bar{Q}(p)$, giving Laplace space drawdown as

$$\bar{s} = \bar{Q}(p) f(r_D, z_D, p_D). \tag{2}$$

Simple Representation of the Piecewise-Linear Pumping Rates

Consider pumping rate history recorded as $Q_0, Q_1, Q_2, \ldots, Q_n$ at discrete time intervals $t_0, t_1, t_2, \ldots, t_n$. Expressing the pumping rate variation as a piecewise-linear function allows writing $Q(t)$ as

$$Q(t) = \sum_{i=1}^{n} \{Q_{i-1} + \beta_i (t - t_{i-1})\}(H_{t_{i-1}} - H_{t_i}) \tag{3}$$

where $\beta_i = (Q_i - Q_{i-1})/(t_i - t_{i-1})$ is the slope of ith linear pumping element and H_{t_i} is unit step function which equals one when $t \geq t_i$ and remains zero elsewhere. Using Laplace transform relations $L\{H_{t_i}\} = \frac{1}{p}\mathrm{e}^{-t_i p}$ and $L\{tf(t)\} = -\frac{d}{dp}F(p)$, where $F(p)$ is the Laplace transform of $f(t)$, the Laplace transform of Equation 3 is given as

$$\bar{Q}(p) = \frac{1}{p} \sum_{i=1}^{n} \left\{ Q_{i-1} + \frac{\beta_i}{p} \right\} (\mathrm{e}^{-t_{i-1} p} - \mathrm{e}^{-t_i p}) - \frac{1}{p} \sum_{i=1}^{n} \beta_i (t_i - t_{i-1}) \mathrm{e}^{-t_i p}. \tag{4}$$

Substituting the Laplace transformed piecewise-linear pumping rate $\bar{Q}(p)$ in Equation 2 gives the Laplace transformed drawdown at any location. The solution corresponding to Equation 2 in the time domain, $s(r, z, t)$, is obtained through numerical inversion of the Laplace transform by means of an algorithm due to Crump (1976) as modified by de Hoog et al. (1982).

To demonstrate the validity of the proposed approach, consider a sinusoidal pumping rate $Q(t) = 2.0 + \sin(30t/\pi)$ m^3/d which has Laplace transform of $\bar{Q}(p) = 2.0/p + \frac{30/\pi}{(30/\pi)^2 + p^2}$. Figure 1 presents the pumping rate variation and drawdown at 1.0 m from a fully penetrating pumping well of zero radius in an isotropic uniform aquifer with $T = 10\mathrm{m}^2/\mathrm{d}$ and $S = 1.0 \times 10^{-5}$. Figure 1 compares drawdown computed directly using the Laplace transform of sinusoidal pumping rate variation (red curve) and the drawdown computed by fitting a piecewise-linear function (blue curve) between pumping rates at every 2 h (black line). The close correspondence between analytically

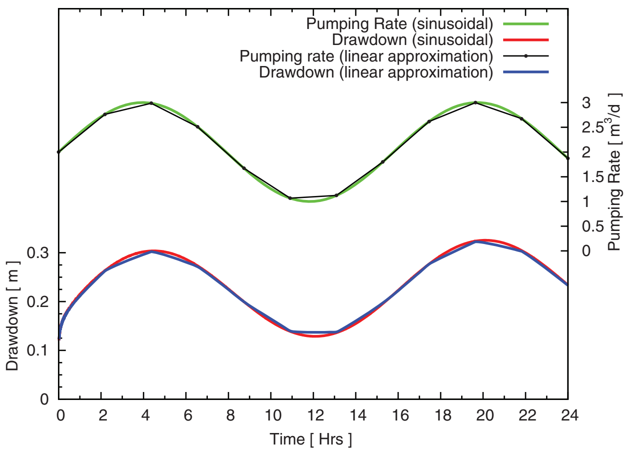

Figure 1. Comparison of analytically evaluated drawdown due to sinusoidal pumping rate variation (green) with (blue) and without (red) piecewise-linear approximations.

computed drawdowns with and without piecewise-linear approximations validates the proposed methodology.

However, the simple piecewise-linear representation of the variable pumping rates presented in Equation 4 does not always produce satisfactory results. Consider hypothetical pumping test where pumping rate varies rapidly (shown in green lines in Figure 2). Figure 2 also shows the computed drawdown at a point located 1.0 m from the fully penetrating pumping will of zero radius in the same isotropic and uniform aquifer. The drawdown computed using piecewise-linear approximation (red lines) shows oscillatory instability (Gibb's effect) near the time where the abrupt change in pumping rate (slope of linear element $\beta_i \to \infty$) occurs; for example, this can occur if the pumping is discontinued abruptly.

Convoluted Representation of the Piecewise-Linear Pumping Rates

To avoid such numerical instabilities, we implemented a convolution method based on a linear

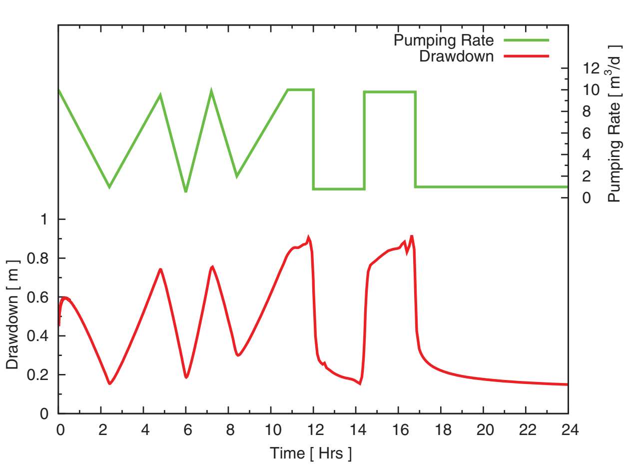

Figure 2. Drawdown evaluated using simple representation of piecewise-linear variation of pumping rate (red curve).

combination of pumping and injection events, which is analogous to the convolution used for the constant-rate step approach. Each period of linear pumping rate change can be decomposed into a combination of linear pumping and injection events. Using a discrete convolution integral resulting drawdown is expressed as

$$s(t) = \sum_{i=1}^{n} \{s_{a'}(t - t_{i-1}) + s_{b'}(t - t_i)\} \tag{5}$$

where $s_{a'}(t)$ and $s_{b'}(t)$ are time-domain drawdown due to pumping $Q_{a'}(t) = Q_{i-1} + \beta_i t$ and injection $Q_{b'}(t) = -Q_i - \beta_i t$ with corresponding Laplace transforms $\bar{Q}_{a'}(p) = \frac{1}{p}\left(Q_{i-1} + \frac{\beta_i}{p}\right)$ and $\bar{Q}_{b'}(p) = \frac{-1}{p}\left(Q_i + \frac{\beta_i}{p}\right)$ respectively.

As shown in Figure 3, numerical instabilities observed when a simple piecewise-linear approach of representing pumping rate variation is applied (red line; Equation 4) can be entirely avoided by applying the method based on convolution of a linear set of pumping and injection events (blue line). The code WELLS has implementation of both simple and convoluted schemes.

Synthetic Example

Consider a 7-m-thick isotropic confined aquifer ($K_D = 1.0$) with horizontal hydraulic conductivity $K_r = 5.01$ m/d and specific storage $S_s = 5.01 \times 10^{-6}\text{m}^{-1}$. A pumping well of infinitesimal diameter penetrates the upper 3.5 m of the confined aquifer and discharges at variable rate. The pumping rate is assumed to vary linearly and the changes occur at every hour as shown in Figure 4 (blue stars). Drawdowns were simulated at over 1000 temporal values uniformly spaced in log space spanning from 10^{-4} to 1.0 d. To mimic condition during an actual test, a random noise of ±5% magnitude was added to the recorded drawdowns. The goal of the synthetic test

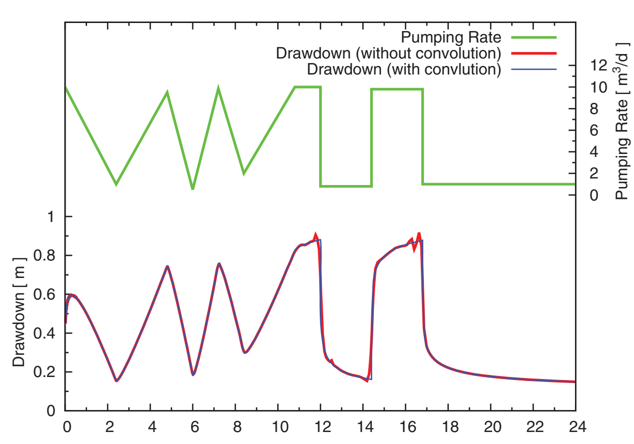

Figure 3. Comparison of drawdown evaluated using simple representation of piecewise-linear variation of pumping rate (red curve) with the drawdown evaluated using convoluted representation of piecewise-linear pumping rate variation (blue curve).

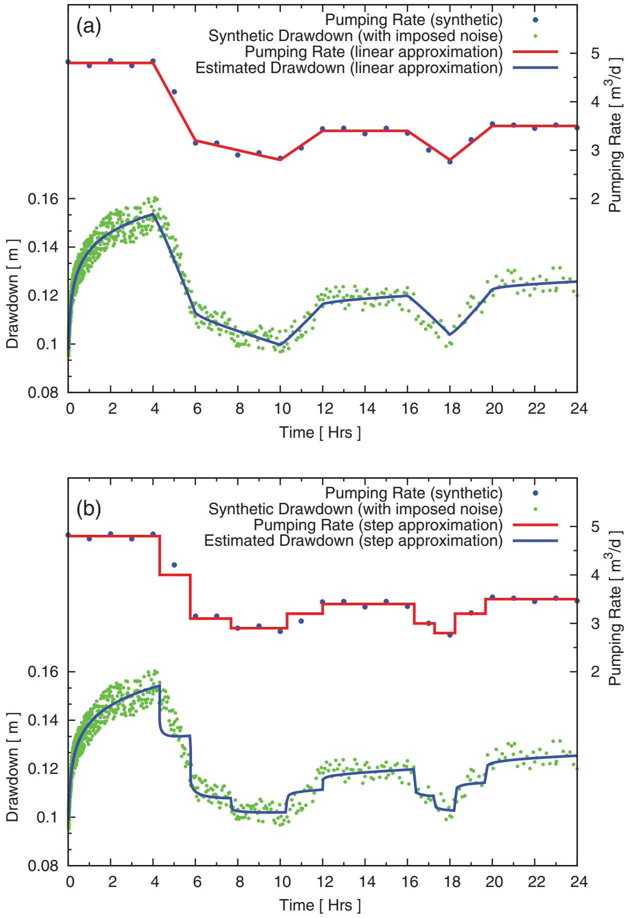

Figure 4. Comparison of inversely estimated drawdown (blue line) with synthetic drawdown (green dots) when pumping rate variation (blue dots) are approximated by (a) equivalent linear changes (red line) and (b) step changes (red line).

analysis is to estimate the aquifer parameters based on the pumping test data applying two different approaches to characterize pumping rate variability: (1) the piecewise-linear approach proposed here and (2) piecewise-constant step approach. The two approximations of the pumping rate variability are presented in Figure 4 (red lines). The approximate pumping rates are adjusted so that the total amount of water pumped during the pumping test in both cases is the same. Note that, the pumping rate is represented by 8 piecewise-linear regions and 10 steps of constant pumping rate.

The parameters were then inversely estimated by minimizing the sum of squared difference between model predicted drawdowns and synthetic drawdowns using the PEST code Doherty (1994) for the two approximation of the variable pumping rate. Figure 4 compares the best fit model predicted drawdown with synthetic drawdown, and Table 1 lists the estimated parameters. Table 1 also lists the percentage error in the estimated parameters (values in closed brackets) and sum of squared error (SSE) in estimated drawdown. The piecewise-linear approximation improves the hydraulic conductivity estimates by a factor of about 3 and specific storage by a factor of about 2, it also results in better representation of the actual drawdowns observed during the pumping test (based on a comparison of simulated drawdowns presented in Figure 4). This demonstrates that for the cases where pumping rate variations are better represented by piecewise-linear changes, our methodology will result in a better posed problem for parameter estimation. As pumping rate variations may not be adequately represented by a relatively small number of constant-rate step changes, the piecewise-constant approach for representing pumping transients is not always sufficient to accurately represent the observed drawdowns and reliably estimate aquifer parameters.

Table 1
Comparison of Estimated Parameters and Sum of Squared Errors (SSE) in Estimated Drawdowns with the Synthetic True Case (Column 2) When Time Varying Pumping Rate Is Approximated as Piecewise-Linear (Column 3) and Step Function (Column 4)

Quantity	True	Linear Changes	Step Changes
K_r [m/d]	5.01	5.04 (0.60%)	5.10 (1.79%)
$S_s \times 10^{-6}$m^{-1}	5.01	4.53 (9.58%)	3.99 (20.36%)
SSE m^2	—	1.65×10^{-2}	1.84×10^{-1}

Summary and Conclusions

The piecewise-linear approximation of time-varying pumping rate is implemented for confined, unconfined, and leaky aquifers in the computer program WELLS (http://wells.lanl.gov) for multi-well variable-rate analysis of pumping test data. The piecewise-linear approximation can represent fairly well any time-varying pumping rates and can reproduce the drawdown for sinusoidal tests with a relatively small number of piecewise-linear discretization sections. For cases when the slope of the linear pumping event is very large (i.e., $\beta_i \to \infty$), the discrete convolution integral approach can be applied to superimpose a combination of pumping and injection steps and thereby avoid instabilities in the numerical Laplace inversion. Our synthetic test case demonstrates that the piecewise-linear approximation can reduce the uncertainty associated with parameter estimation by providing a better representation of varying pumping rates.

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